

K-THEORY, LOCAL COHOMOLOGY AND TANGENT SPACES TO HILBERT SCHEMES

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ABSTRACT. By using K-theory, we construct a map from tangent spaces to Hilbert schemes to local cohomology groups: $\pi : T_Y \text{Hilb}^q(X) \rightarrow H_y^q(\Omega_{X/\mathbb{Q}}^{q-1})$. And we use this map π to answer affirmatively(after slight modification) a question by Mark Green and Phillip Griffiths in [4] on constructing a map from tangent spaces to Hilbert schemes $T_Y \text{Hilb}^q(X)$ to those to cycle class groups $TZ^q(X)$, see Question 1.2.

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1. Introduction

Let X be a smooth projective variety over a field k of characteristic 0 and let $Y \subset X$ be a subvariety(not necessarily locally complete intersection) of codimension q , with generic point y . Considering Y as an element of $\text{Hilb}^q(X)$, it is well known that the Zariski tangent space $T_Y \text{Hilb}^q(X)$ can be identified with $H^0(Y, \mathcal{N}_{Y/X})$, where $\mathcal{N}_{Y/X}$ is the normal sheaf.

Y can be also considered an element of the cycle class group $Z^q(X)$ and we are interested in defining the tangent space $TZ^q(X)$ to the cycle class group $Z^q(X)$. In [4], Mark Green and Phillip Griffiths define $TZ^q(X)$ for $q = 1$ and $q = \dim(X)$ and leaves the general case as an open question. In [9], we define $TZ^q(X)$ for any integer q , generalizing Green and Griffiths' definitions. We recall the following fact from [9] for our purpose, and refer to [4, 9] for definitions of $TZ^q(X)$.

2010 *Mathematics Subject Classification.* 14C25.

Theorem 1.1 (Theorem 3.3 in [9]). *For any integer q , the tangent space $TZ^q(X)$ is identified with $\text{Ker}(\partial_1^{q,-q})$:*

$$TZ^q(X) \cong \text{Ker}(\partial_1^{q,-q}),$$

where $\partial_1^{q,-q}$ is the differential of the Cousin complex of $\Omega_{X/\mathbb{Q}}^{q-1}$ in position q :

$$0 \rightarrow \Omega_{k(X)/\mathbb{Q}}^{q-1} \rightarrow \cdots \rightarrow \bigoplus_{y \in X^{(q)}} H_y^q(\Omega_{X/\mathbb{Q}}^{q-1}) \xrightarrow{\partial_1^{q,-q}} \bigoplus_{x \in X^{(q+1)}} H_x^{q+1}(\Omega_{X/\mathbb{Q}}^{q-1}) \rightarrow \cdots$$

Now, we want to compare the relation between $T_Y\text{Hilb}^q(X)$ and $TZ^q(X)$. The following question is suggested by Mark Green and Phillip Griffiths in [4](see page 18 and page 87-89):

Question 1.2. [4] *Is it possible to define a map from tangent spaces to Hilbert schemes to those to cycles class groups:*

$$T_Y\text{Hilb}^q(X) \rightarrow TZ^q(X)?$$

For $q = \dim(X)$, this has been answered affirmatively by Green and Griffiths in [4], see Section 7.2 for details:

Theorem 1.3. [4] *For $q = \dim(X)$, there exists a map from tangent space to Hilbert scheme to that to cycles class groups*

$$T_Y\text{Hilb}^q(X) \rightarrow TZ^q(X).$$

The main results of this short note is to construct a map from $T_Y\text{Hilb}^q(X)$ to $H_y^q(\Omega_{X/\mathbb{Q}}^{q-1})$ in Definition 4.1 and use this map to answer affirmatively(after slight modification) the above Question 1.2 by Mark Green and Phillip Griffiths, see Theorem 4.6.

Acknowledgments This short note is a follow-up of [9]. The author is very grateful to Mark Green and Phillip Griffiths for asking interesting questions. He is also very grateful to Spencer Bloch and Christophe Soulé for sharing their ideas.

The author also thanks the following professors for discussions: Ben Dribus, David Eisenbud, Jerome Hoffman, Luc Illusie, Chao Zhang.

Notations and conventions.

(1).K-theory used in this note will be Thomason-Trobaugh non-connective K-theory, if not stated otherwise.

(2).For any abelian group M , $M_{\mathbb{Q}}$ denotes the image of M in $M \otimes_{\mathbb{Z}} \mathbb{Q}$.

(3). $X[\varepsilon]$ is the first order trivial deformation of X , i.e., $X[\varepsilon] = X \times_k \text{Spec}(k[\varepsilon])$, where $\text{Spec}(k[\varepsilon])$ denotes the dual number, $\varepsilon^2 = 0$.

2. K-theory and tangent spaces to Hilbert schemes

Let X be a smooth projective variety over a field k of characteristic 0 and let $Y \subset X$ be a subvariety (not necessarily locally complete intersection) of codimension q . Let $i : Y \rightarrow X$ be the inclusion, then i_*O_Y is a coherent O_X -module and can be resolved by a bounded complex of vector bundles on X . Let Y' be a first order deformation of Y , that is, $Y' \subset X[\varepsilon]$ such that Y' is **flat** over $\text{Spec}(k[\varepsilon])$ and $Y' \otimes_{k[\varepsilon]} k \cong Y$.

To fix notations, let y be the generic point of Y and let \mathcal{I}_Y be the ideal sheaf of Y . We have the following short exact sequence:

$$0 \rightarrow \mathcal{I}_Y \rightarrow O_X \rightarrow i_*O_Y \rightarrow 0,$$

whose localization at y is the short exact sequence:

$$0 \rightarrow (\mathcal{I}_Y)_y \rightarrow O_{X,y} \rightarrow (i_*O_Y)_y \rightarrow 0.$$

We have $O_{Y,y} = O_{X,y}/(\mathcal{I}_Y)_y$. Since $O_{Y,y}$ is a field, so $(\mathcal{I}_Y)_y$ is the maximal ideal of $O_{X,y}$. Since $O_{X,y}$ is a regular local ring with dimension q , so the maximal ideal $(\mathcal{I}_Y)_y$ is generated by a regular sequence of length q : f_1, \dots, f_q .

Let $\mathcal{I}_{Y'}$ be the ideal sheaf of Y' , then $\mathcal{I}_{Y'}/(\varepsilon)\mathcal{I}_{Y'} = \mathcal{I}_Y$ because of flatness. So we have $(\mathcal{I}_{Y'})_y/(\varepsilon)(\mathcal{I}_{Y'})_y = (\mathcal{I}_Y)_y$. Lift f_1, \dots, f_q to $f_1 + \varepsilon g_1, \dots, f_q + \varepsilon g_q$ in $(\mathcal{I}_{Y'})_y$, where $g_1, \dots, g_q \in O_{X,y}$, then $f_1 + \varepsilon g_1, \dots, f_q + \varepsilon g_q$ generates $(\mathcal{I}_{Y'})_y$ because of Nakayama's lemma:

$$(\mathcal{I}_{Y'})_y = (f_1 + \varepsilon g_1, \dots, f_q + \varepsilon g_q).$$

Moreover, $f_1 + \varepsilon g_1, \dots, f_q + \varepsilon g_q$ is a regular sequence which can be checked directly.

We use $F_\bullet(f_1 + \varepsilon g_1, \dots, f_q + \varepsilon g_q)$ to denote the Koszul complex associated to the regular sequence $f_1 + \varepsilon g_1, \dots, f_q + \varepsilon g_q$, which is a resolution of $O_{X,y}[\varepsilon]/(f_1 + \varepsilon g_1, \dots, f_q + \varepsilon g_q)$:

$$0 \longrightarrow F_q \xrightarrow{A_q} F_{q-1} \xrightarrow{A_{q-1}} \dots \xrightarrow{A_2} F_1 \xrightarrow{A_1} F_0,$$

where each $F_i = \bigwedge^i O_{X,y}^{\oplus q}[\varepsilon]$ and $A_i : \bigwedge^i O_{X,y}^{\oplus q}[\varepsilon] \rightarrow \bigwedge^{i-1} O_{X,y}^{\oplus q}[\varepsilon]$ are defined as usual.

Recall that Milnor K-groups with support are rationally defined in terms of eigenspaces of Adams operations in [8].

Definition 2.1. [8] *Let X be an finite equi-dimensional noetherian scheme and $x \in X$ satisfy $\dim O_{X,x} = j$, for any integer m , Milnor K-group with support $K_m^M(O_{X,x} \text{ on } x)$ is defined to be*

$$K_m^M(O_{X,x} \text{ on } x) := K_m^{(m+j)}(O_{X,x} \text{ on } x)_{\mathbb{Q}},$$

where $K_m^{(m+j)}$ is the eigenspace for $\psi^k = k^{m+j}$ and ψ^k is the Adams operations.

Theorem 2.2 (Prop 4.12 of [3]). *The Adams operations ψ^k defined on perfect complexes, defined by Gillet-Soulé in [3], satisfy $\psi^k(F_\bullet(f_1 + \varepsilon g_1, \dots, f_q + \varepsilon g_q)) = k^q F_\bullet(f_1 + \varepsilon g_1, \dots, f_q + \varepsilon g_q)$.*

Hence, $F_\bullet(f_1 + \varepsilon g_1, \dots, f_q + \varepsilon g_q)$ is of eigenweight q and can be considered as an element of $K_0^{(q)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}}$:

$$F_\bullet(f_1 + \varepsilon g_1, \dots, f_q + \varepsilon g_q) \in K_0^{(q)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}} = K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]).$$

Definition 2.3. *We define a map $\mu : H^0(Y, \mathcal{N}_{Y/X}) \rightarrow K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])$ as follows:*

$$\begin{aligned} \mu : H^0(Y, \mathcal{N}_{Y/X}) &\rightarrow K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \\ Y' &\longrightarrow F_\bullet(f_1 + \varepsilon g_1, \dots, f_q + \varepsilon g_q). \end{aligned}$$

3. Chern character

For any integer m , let $K_m^{(i)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon)_{\mathbb{Q}}$ denote the weight i eigenspace of the relative K-group, that is, the kernel of the natural projection

$$K_m^{(i)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}} \xrightarrow{\varepsilon=0} K_m^{(i)}(O_{X,y} \text{ on } y)_{\mathbb{Q}}.$$

Recall that we have proved the following isomorphisms in [8]:

Theorem 3.1 (Corollary 3.11 in [8]). *Let X be a smooth projective variety over a field k of characteristic 0 and let $y \in X^{(q)}$. Chern character induces the following isomorphisms between relative K-groups and local cohomology groups:*

$$K_m^{(i)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon)_{\mathbb{Q}} \cong H_y^q(\Omega_{O_{X,y}/\mathbb{Q}}^{\bullet, (i)}),$$

where

$$\begin{cases} \Omega_{O_{X,y}/\mathbb{Q}}^{\bullet, (i)} &= \Omega_{O_{X,y}/\mathbb{Q}}^{2i-(m+q)-1}, \text{ for } \frac{m+q}{2} < i \leq m+q, \\ \Omega_{O_{X,y}/\mathbb{Q}}^{\bullet, (i)} &= 0, \text{ else.} \end{cases}$$

Let $K_m^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon)$ denote the relative K-group, that is, the kernel of the natural projection

$$K_m^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \xrightarrow{\varepsilon=0} K_m^M(O_{X,y} \text{ on } y).$$

In other words, $K_m^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon)$ is the relative K-group $K_m^{(m+q)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon)_{\mathbb{Q}}$. For our purpose, in particular, by taking $i = q$ and $m = 0$ in Theorem 3.1, we obtain the following formula:

Corollary 3.2.

$$K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon) \cong H_y^q(\Omega_{O_{X,y}/\mathbb{Q}}^{q-1}).$$

Definition 3.3. (relative) Chern character induces the following natural surjective maps

$$\text{Chern} : K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \rightarrow H_y^q(\Omega_{X/\mathbb{Q}}^{q-1}).$$

Angénoil and Lejeune-Jalabert's construction of Chern character Now, We recall a beautiful construction of Angénoil and Lejeune-Jalabert which describes the Chern character in Definition 3.3

$$\text{Chern} : K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \rightarrow H_y^q(\Omega_{X/\mathbb{Q}}^{q-1}).$$

Any element $M \in K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \subset K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}}$ is represented by a strict perfect complex L_{\bullet} supported at $y[\varepsilon]$:

$$0 \longrightarrow F_n \xrightarrow{M_n} F_{n-1} \xrightarrow{M_{n-1}} \dots \xrightarrow{M_2} F_1 \xrightarrow{M_1} F_0 \longrightarrow 0,$$

where each $F_i = O_{X,y}[\varepsilon]^{r_i}$ and M_i 's are matrices with entries in $O_{X,y}[\varepsilon]$.

Definition 3.4 (Page 24 in [1]). *The local fundamental class attached to this perfect complex is defined to be the following collection:*

$$[L_{\bullet}]_{loc} = \left\{ \frac{1}{q!} dM_i \circ dM_{i+1} \circ \dots \circ dM_{i+q-1} \right\}, i = 0, 1, \dots$$

where $d = d_{/\mathbb{Q}}$ and each dM_i is the matrix of absolute differentials. In other word,

$$dM_i \in \text{Hom}(F_i, F_{i-1} \otimes \Omega_{O_{X,y}[\varepsilon]/\mathbb{Q}}^1).$$

Theorem 3.5 (Lemme 3.1.1 on page 24 and Def 3.4 on page 29 in [1]). $[L_{\bullet}]_{loc}$ defined above is a cycle in $\mathcal{H}om(L_{\bullet}, \Omega_{O_{X,y}[\varepsilon]/\mathbb{Q}}^q \otimes L_{\bullet})$, and the image of $[L_{\bullet}]_{loc}$ in $H^q(\mathcal{H}om(L_{\bullet}, \Omega_{O_{X,y}[\varepsilon]/\mathbb{Q}}^q \otimes L_{\bullet}))$ does not depend on the choice of the basis of L_{\bullet} .

Since

$$H^q(\mathcal{H}om(L_{\bullet}, \Omega_{O_{X,y}[\varepsilon]/\mathbb{Q}}^q \otimes L_{\bullet})) = \mathcal{E}XT^q(L_{\bullet}, \Omega_{O_{X,y}[\varepsilon]/\mathbb{Q}}^q \otimes L_{\bullet}),$$

the above construction local fundamental class $[L_{\bullet}]_{loc}$ defines an element in $\mathcal{E}XT^q(L_{\bullet}, \Omega_{O_{X,y}[\varepsilon]/\mathbb{Q}}^q \otimes L_{\bullet})$:

$$[L_{\bullet}]_{loc} \in \mathcal{E}XT^q(L_{\bullet}, \Omega_{O_{X,y}[\varepsilon]/\mathbb{Q}}^q \otimes L_{\bullet}).$$

Noting L_{\bullet} is supported on y (same underlying space as $y[\varepsilon]$), there exists the following trace map, see page 98-99 of [1] for more details,

$$\text{Tr} : \mathcal{E}XT^q(L_{\bullet}, \Omega_{O_{X,y}[\varepsilon]/\mathbb{Q}}^q \otimes L_{\bullet}) \longrightarrow H_y^q(\Omega_{X[\varepsilon]/\mathbb{Q}}^q).$$

Definition 3.6 (Definition 2.3.2 on page 99 in [1]). *The image of $[L_\bullet]_{loc}$ under the above trace map is called Newton class, and is denoted by $\mathcal{V}_{L_\bullet}^q$.*

Theorem 3.7 (Lemme 3.2.1 on page 26 and Prop 4.3.1 on page 113 in [1]). *The Newton class $\mathcal{V}_{L_\bullet}^q$ is well-defined on $K_0(O_{X,y}[\varepsilon])$ on $y[\varepsilon]$.*

The truncation map $\lfloor \frac{\partial}{\partial \varepsilon} \rfloor_{\varepsilon=0}: \Omega_{X[\varepsilon]/\mathbb{Q}}^q \rightarrow \Omega_{X/\mathbb{Q}}^{q-1}$ induces a map

$$\begin{aligned} \lfloor \frac{\partial}{\partial \varepsilon} \rfloor_{\varepsilon=0}: H_y^q(\Omega_{X[\varepsilon]/\mathbb{Q}}^q) &\longrightarrow H_y^q(\Omega_{X/\mathbb{Q}}^{q-1}) \\ \mathcal{V}_{L_\bullet}^q &\longrightarrow \mathcal{V}_{L_\bullet}^q \lfloor \frac{\partial}{\partial \varepsilon} \rfloor_{\varepsilon=0}. \end{aligned}$$

Theorem 3.8. *With the above notations, the Chern character map*

$$\text{Chern} : K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \rightarrow H_y^q(\Omega_{X/\mathbb{Q}}^{q-1})$$

can be described as a composition

$$\begin{aligned} K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) &\rightarrow \mathcal{EXT}^q(L_\bullet, \Omega_{O_{X,y}[\varepsilon]/\mathbb{Q}}^q \otimes L_\bullet) \rightarrow H_y^q(\Omega_{X[\varepsilon]/\mathbb{Q}}^q) \rightarrow H_y^q(\Omega_{X/\mathbb{Q}}^{q-1}) \\ L_\bullet &\longrightarrow [L_\bullet]_{loc} \longrightarrow \mathcal{V}_{L_\bullet}^q \longrightarrow \mathcal{V}_{L_\bullet}^q \lfloor \frac{\partial}{\partial \varepsilon} \rfloor_{\varepsilon=0}. \end{aligned}$$

In particular, for the Koszul complex $F_\bullet(f_1 + \varepsilon g_1, \dots, f_q + \varepsilon g_q)$ in Definition 2.3, the above algorithm can be further simplified. To be precise, the following diagram,

$$\begin{cases} F_\bullet(f_1 + \varepsilon g_1, \dots, f_q + \varepsilon g_q) &\longrightarrow O_{X,y}[\varepsilon]/(f_1 + \varepsilon g_1, \dots, f_q + \varepsilon g_q) \\ F_q(\cong O_{X,y}[\varepsilon]) &\xrightarrow{[F_\bullet]_{loc}} F_0 \otimes \Omega_{O_{X,y}[\varepsilon]/\mathbb{Q}}^q (\cong \Omega_{O_{X,y}[\varepsilon]/\mathbb{Q}}^q), \end{cases}$$

where $[F_\bullet]_{loc}$ is short for the local fundamental class attached to $F_\bullet(f_1 + \varepsilon g_1, \dots, f_q + \varepsilon g_q)$, gives an element in $\text{Ext}_{O_{X,y}[\varepsilon]}^q(O_{X,y}[\varepsilon]/(f_1 + \varepsilon g_1, \dots, f_q + \varepsilon g_q), \Omega_{O_{X,y}[\varepsilon]/\mathbb{Q}}^q)$. This in turn gives an element in $H_y^q(\Omega_{X[\varepsilon]/\mathbb{Q}}^q)$, denoted by $\mathcal{V}_{F_\bullet}^q$.

We use $F_\bullet(f_1, \dots, f_q)$ to denote the Koszul complex associated to the regular sequence f_1, \dots, f_q , which is a resolution of $O_{X,y}/(f_1, \dots, f_q)$:

$$0 \longrightarrow F_q \xrightarrow{B_q} F_{q-1} \xrightarrow{B_{q-1}} \dots \xrightarrow{B_2} F_1 \xrightarrow{B_1} F_0.$$

The truncation of $\mathcal{V}_{F_\bullet}^q$ in ε produces an element in $H_y^q(\Omega_{X/\mathbb{Q}}^{q-1})$, which can be represented by the following diagram

$$\begin{cases} F_\bullet(f_1, \dots, f_q) &\longrightarrow O_{X,y}/(f_1, \dots, f_q) \\ F_q(\cong O_{X,y}) &\xrightarrow{[F_\bullet]_{loc} \lfloor \frac{\partial}{\partial \varepsilon} \rfloor_{\varepsilon=0}} F_0 \otimes \Omega_{O_{X,y}/\mathbb{Q}}^{q-1} (\cong \Omega_{O_{X,y}/\mathbb{Q}}^{q-1}). \end{cases}$$

Further concrete examples can be found in Chap 7 of [4](page 90-91).

4. The map π

Definition 4.1. We define a map from $T_Y \text{Hilb}^q(X)$ to $H_y^q(\Omega_{X/\mathbb{Q}}^{q-1})$ by composing Chern character in Definition 3.3 with μ in Definition 2.3:

$$\pi : T_Y \text{Hilb}^q(X) \xrightarrow{\mu} K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \xrightarrow{\text{Chern}} H_y^q(\Omega_{X/\mathbb{Q}}^{q-1}).$$

Recall that the Cousin complex of $\Omega_{X/\mathbb{Q}}^{q-1}$ is of the form:

$$0 \rightarrow \Omega_{k(X)/\mathbb{Q}}^{q-1} \rightarrow \cdots \rightarrow \bigoplus_{y \in X^{(q)}} H_y^q(\Omega_{X/\mathbb{Q}}^{q-1}) \xrightarrow{\partial_1^{q,-q}} \bigoplus_{x \in X^{(q+1)}} H_x^{q+1}(\Omega_{X/\mathbb{Q}}^{q-1}) \rightarrow \cdots$$

and the tangent space $TZ^q(X)$ is identified with $\text{Ker}(\partial_1^{q,-q})$, see Theorem 1.1.

For $q = \dim(X)$, $\partial_1^{q,-q} = 0$ because of dimensional reason. So $TZ^q(X) = \text{Ker}(\partial_1^{q,-q}) = \bigoplus_{y \in X^{(q)}} H_y^q(\Omega_{X/\mathbb{Q}}^{q-1})$.

Corollary 4.2. For $q = \dim(X)$, the map π defines a map from $T_Y \text{Hilb}^q(X)$ to $TZ^q(X)$ and it agrees with the map by Green and Griffiths in Theorem 1.3.

We want to know, for general q , whether this map π defines a map from $T_Y \text{Hilb}^q(X)$ to $TZ^q(X)$, as Green and Griffiths asked in Question 1.2.

Remark 4.3. In an E-mail to the author, Christophe Soulé teaches him to consider the product of Koszul complex. This guides us to the following example, showing π doesn't define a map from $T_Y \text{Hilb}^q(X)$ to $TZ^q(X)$ in general. The Koszul technique is also used in Theorem 4.6.

The author sincerely thanks Christophe Soulé for very helpful suggestions.

Example 4.4 (should already be known to experts). Let X be a smooth projective surface over a field k of characteristic 0. Let Y_1 and Y_2 be two curves on X with generic point y_1 and y_2 respectively. For simplicity, we work locally in Zariski topology and assume Y_1 and Y_2 intersect transversely at a point x . We further assume, around x , Y_1 and Y_2 are defined by f_1 and f_2 respectively. One has

$$O_{X,y_1} = (O_{X,x})_{(f_1)}; \quad O_{X,y_2} = (O_{X,x})_{(f_2)}.$$

We consider the infinitesimal deformation Y'_1 of Y_1 which is generically given by $f_1 + \varepsilon \frac{1}{f_2}$, where $\frac{1}{f_2} \in O_{X,y_1} = (O_{X,x})_{(f_1)}$ (Note $\frac{1}{f_2} \notin O_{X,x}$).

The Koszul complex is of the form

$$(O_{X,x}[\varepsilon])_{(f_1)} \xrightarrow{f_1 + \varepsilon \frac{1}{f_2}} (O_{X,x}[\varepsilon])_{(f_1)}.$$

Then $\pi(Y'_1)$ is represented by the following diagram,

$$\begin{cases} (O_{X,x})_{(f_1)} \xrightarrow{f_1} (O_{X,x})_{(f_1)} \longrightarrow (O_{X,x})_{(f_1)}/(f_1) \longrightarrow 0 \\ (O_{X,x})_{(f_1)} \xrightarrow{\frac{1}{f_2}} (O_{X,x})_{(f_1)}, \end{cases}$$

and $\partial_1^{1,-1}(\pi(Y'_1))$ in $H_x^2(O_X)$ is represented by the following diagram:

$$\begin{cases} O_{X,x} \xrightarrow{(f_2, -f_1)^T} O_{X,x}^{\oplus 2} \xrightarrow{(f_1, f_2)} O_{X,x} \longrightarrow O_{X,x}/(f_1, f_2) \longrightarrow 0 \\ O_{X,x} \xrightarrow{1} O_{X,x}, \end{cases}$$

which is certainly not zero.

This trivial example shows that the image of π doesn't necessarily lie in $TZ^q(X)$ (the kernel of $\partial_1^{q,-q}$). However, we will show that, in Theorem 4.6 below, there exists $Z \subset X$ of codimension q and exists $Z' \in T_Z \text{Hilb}^q(X)$ such that $\pi(Y') + \pi(Z') \in TZ^q(X)$.

To fix notations, let X be a smooth projective variety over a field k of characteristic 0 and let $Y \subset X$ be a subvariety (not necessarily locally complete intersection) of codimension q , with generic point y . Let $W \subset Y$ be a subvariety of codimension 1 in Y , with generic point w . One assumes W is generically defined by $f_1, f_2, \dots, f_q, f_{q+1}$ and Y is generically defined by f_1, f_2, \dots, f_q . So one has $O_{X,y} = (O_{X,w})_P$, where P is the idea $(f_1, f_2, \dots, f_q) \subset O_{X,w}$.

We assume Y' is generically given by $(f_1 + \varepsilon g_1, f_2, \dots, f_q)$, where $g_1 \in O_{X,y} = (O_{X,w})_P$. So we can write $g_1 = \frac{a}{b}$, where $a, b \in O_{X,w}$ and $b \notin P$. Then b is either in or not in the maximal idea $(f_1, f_2, \dots, f_q, f_{q+1}) \subset O_{X,w}$.

Lemma 4.5. *If $b \notin (f_1, f_2, \dots, f_q, f_{q+1})$, then $\partial_1^{q,-q}(\pi(Y')) = 0$.*

Proof. If $b \notin (f_1, f_2, \dots, f_q, f_{q+1})$, then b is a unit in $O_{X,w}$, this says $g_1 = \frac{a}{b} \in O_{X,w}$. Then $\pi(Y')$ is represented by the following diagram

$$\begin{cases} F_\bullet(f_1, f_2, \dots, f_q) \longrightarrow (O_{X,w})_P/(f_1, f_2, \dots, f_q) \\ F_q(\cong (O_{X,w})_P) \xrightarrow{g_1 df_2 \wedge \dots \wedge df_q} F_0 \otimes \Omega_{(O_{X,w})_P/\mathbb{Q}}^{q-1} (\cong \Omega_{(O_{X,w})_P/\mathbb{Q}}^{q-1}). \end{cases}$$

Since $f_{q+1} \notin P$, f_{q+1}^{-1} exists in $(O_{X,w})_P$, we can write $g_1 df_2 \wedge \dots \wedge df_q = \frac{g_1 f_{q+1}}{f_{q+1}} df_2 \wedge \dots \wedge df_q$. $\partial_1^{q,-q}(\pi(Y'))$ is represented by the following

diagram

$$\begin{cases} F_{\bullet}(f_1, f_2, \dots, f_q, f_{q+1}) & \longrightarrow & O_{X,w}/(f_1, f_2, \dots, f_q, f_{q+1}) \\ F_{q+1}(\cong O_{X,w}) & \xrightarrow{g_1 f_{q+1} df_2 \wedge \dots \wedge df_q} & F_0 \otimes \Omega_{O_{X,w}/\mathbb{Q}}^{q-1} (\cong \Omega_{O_{X,w}/\mathbb{Q}}^{q-1}). \end{cases}$$

Noting $g_1 f_{q+1} df_2 \wedge \dots \wedge df_q \equiv 0 \in \text{Ext}_{O_{X,w}}^{q+1}(O_{X,w}/(f_1, f_2, \dots, f_q, f_{q+1}), \Omega_{O_{X,w}/\mathbb{Q}}^{q-1})$, $\partial_1^{q,-q}(\pi(Y')) = 0$.

□

This lemma doesn't contradict Example 4.4 above, where $f_2 \in (f_1, f_2) \subset O_{X,x}$.

Theorem 4.6. *With notations as above, for $Y' \in T_Y \text{Hilb}^q(X)$ which is generically defined by $(f_1 + \varepsilon g_1, f_2, \dots, f_q)$, where $g_1 = \frac{a}{b} \in O_{X,y} = (O_{X,w})_P$,*

- *Case 1: if $b \notin (f_1, f_2, \dots, f_q, f_{q+1})$, then $\pi(Y') \in TZ^q(X)$, i.e., $\partial_1^{q,-q}(\pi(Y')) = 0$.*
- *Case 2: if $b \in (f_1, f_2, \dots, f_q, f_{q+1})$, there exists $Z \subset X$ of codimension q and exists $Z' \in T_Z \text{Hilb}^q(X)$ such that $\pi(Y') + \pi(Z') \in TZ^q(X)$, i.e., $\partial_1^{q,-q}(\pi(Y') + \pi(Z')) = 0$.*

Proof. Case 1 is Lemma 4.5. Now, we consider the case $b \in (f_1, f_2, \dots, f_q, f_{q+1})$. Since $b \notin (f_1, f_2, \dots, f_q)$, we can write $b = \sum_{i=1}^q a_i f_i^{n_i} + a_{q+1} f_{q+1}^{n_{q+1}}$, where a_{q+1} is a unit in $O_{X,w}$ and each n_j is some integer. For simplicity, we assume each $n_j = 1$ and $a_{q+1} = 1$.

Since Y' is generically given by $(f_1 + \varepsilon g_1, f_2, \dots, f_q)$, then $\pi(Y')$ is represented by the following diagram

$$\begin{cases} F_{\bullet}(f_1, f_2, \dots, f_q) & \longrightarrow & (O_{X,w})_P/(f_1, f_2, \dots, f_q) \\ F_q(\cong (O_{X,w})_P) & \xrightarrow{\frac{a}{b} df_2 \wedge \dots \wedge df_q} & F_0 \otimes \Omega_{(O_{X,w})_P/\mathbb{Q}}^{q-1} (\cong \Omega_{(O_{X,w})_P/\mathbb{Q}}^{q-1}). \end{cases}$$

Noting $\frac{1}{b} - \frac{1}{f_{q+1}} = \frac{-\sum_{i=1}^q a_i f_i}{b f_{q+1}}$, the above diagram representing $\pi(Y')$ can be replaced by the following one

$$\begin{cases} F_{\bullet}(f_1, f_2, \dots, f_q) & \longrightarrow & (O_{X,w})_P/(f_1, f_2, \dots, f_q) \\ F_q(\cong (O_{X,w})_P) & \xrightarrow{\frac{a}{f_{q+1}} df_2 \wedge \dots \wedge df_q} & F_0 \otimes \Omega_{(O_{X,w})_P/\mathbb{Q}}^{q-1} (\cong \Omega_{(O_{X,w})_P/\mathbb{Q}}^{q-1}). \end{cases}$$

Then $\partial_1^{q,-q}(\pi(Y'))$ is represented by the following diagram

$$\begin{cases} F_\bullet(f_1, f_2, \dots, f_q, f_{q+1}) & \longrightarrow & O_{X,w}/(f_1, f_2, \dots, f_q, f_{q+1}) \\ F_{q+1}(\cong O_{X,w}) & \xrightarrow{adf_2 \wedge \dots \wedge df_q} & F_0 \otimes \Omega_{O_{X,w}/\mathbb{Q}}^{q-1} (\cong \Omega_{O_{X,w}/\mathbb{Q}}^{q-1}). \end{cases}$$

Now, we consider the subscheme defined by $(f_{q+1}, f_2, \dots, f_q)$,

$$(4.1) \quad Z := \text{Spec}(O_{X,w}/(f_{q+1}, f_2, \dots, f_q)).$$

Let P' denote the prime $(f_{q+1}, f_2, \dots, f_q) \subset O_{X,w}$ and let z denote the generic point of Z , then one has $O_{X,z} = (O_{X,w})_{P'}$.

Let Z' be a first order infinitesimal deformation of Z , which is generically given by $(f_{q+1} + \varepsilon \frac{a}{f_1}, f_2, \dots, f_q)$.

$\pi(Z')$ is represented by the following diagram:

$$\begin{cases} F_\bullet(f_{q+1}, f_2, \dots, f_q) & \longrightarrow & (O_{X,w})_{P'}/(f_{q+1}, f_2, \dots, f_q) \\ F_q(\cong (O_{X,w})_{P'}) & \xrightarrow{\frac{a}{f_1} df_2 \wedge \dots \wedge df_q} & F_0 \otimes \Omega_{(O_{X,w})_{P'}/\mathbb{Q}}^{q-1} (\cong \Omega_{(O_{X,w})_{P'}/\mathbb{Q}}^{q-1}), \end{cases}$$

and $\partial_1^{q,-q}(\pi(Z'))$ is represented by the following diagram:

$$\begin{cases} F_\bullet(f_{q+1}, f_2, \dots, f_q, f_1) & \longrightarrow & O_{X,w}/(f_{q+1}, f_2, \dots, f_q, f_1) \\ F_{q+1}(\cong O_{X,w}) & \xrightarrow{adf_2 \wedge \dots \wedge df_q} & F_0 \otimes \Omega_{O_{X,w}/\mathbb{Q}}^{q-1} (\cong \Omega_{O_{X,w}/\mathbb{Q}}^{q-1}). \end{cases}$$

Here, $F_\bullet(f_1, f_2, \dots, f_q, f_{q+1})$ and $F_\bullet(f_{q+1}, f_2, \dots, f_q, f_1)$ are Koszul resolutions of $O_{X,w}/(f_1, f_2, \dots, f_q, f_{q+1})$ and $O_{X,w}/(f_{q+1}, f_2, \dots, f_q, f_1)$ and these two Koszul complexes $F_\bullet(f_1, f_2, \dots, f_q, f_{q+1})$ and $F_\bullet(f_{q+1}, f_2, \dots, f_q, f_1)$ are related by the following commutative diagram, see page 691 of [5],

$$\begin{array}{ccccccc} O_{X,w} & \xrightarrow{D_{q+1}} & \wedge^q O_{X,w}^{\oplus q+1} & \xrightarrow{D_q} \dots & \longrightarrow & O_{X,w}^{\oplus q+1} & \xrightarrow{D_1} O_{X,w} \\ \det A_1 \downarrow & & \wedge^q A_1 \downarrow & & \downarrow & A_1 \downarrow & = \downarrow \\ O_{X,w} & \xrightarrow{E_{q+1}} & \wedge^q O_{X,w}^{\oplus q+1} & \xrightarrow{E_q} \dots & \longrightarrow & O_{X,w}^{\oplus q+1} & \xrightarrow{E_1} O_{X,w} \end{array}$$

where each D_i and E_i are defined as usual. In particular, $D_1 = (f_1, f_2, \dots, f_q, f_{q+1})$ and $E_1 = (f_{q+1}, f_2, \dots, f_q, f_1)$. And A_1 is the matrix:

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 1 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Since $\det A_1 = -1$, one has

$$\partial_1^{q,-q}(\pi(Z')) = -\partial_1^{q,-q}(\pi(Y')) \in \text{Ext}_{O_{X,w}}^{q+1}(O_{X,w}/(f_1, f_2, \dots, f_q, f_{q+1}), \Omega_{O_{X,w}/\mathbb{Q}}^{q-1}),$$

consequently, $\partial_1^{q,-q}(\pi(Z') + \pi(Y')) = 0 \in H_w^{q+1}(\Omega_{O_{X,w}/\mathbb{Q}}^{q-1})$. In other words,

$$\pi(Z') + \pi(Y') \in TZ^q(X).$$

□

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